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On Simultaneous Chebyshev Approximation

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Diaz and McLaughlin [1] have recently proved the following

THEOREM A. *For a and b real numbers* $(a < b)$, *let F and f denote real-valued functions defined on the interval* $[a, b]$. *Let S denote a nonempty set of real-valued functions defined on* $[a, b]$. *Then*

$$\inf_{s \in S} \| \, | \tfrac{1}{2}(F + f) - s \, | + \tfrac{1}{2} \, | F - f | \, \| = \inf_{s \in S} \max\{\| F - s \|, \| f - s \|\}, \quad (1)$$

where

$$\| g \| = \sup_{a \leqslant x \leqslant b} | g(x)|.$$

They ask (cf. [1, Sect. 5]) if such a result is also valid for complex-valued functions and observe that if it is, the method of proof cannot be identical to that in the real case since the crucial lemma

$$| \tfrac{1}{2}(m + n)| + | \tfrac{1}{2}(m - n)| = \max(| m |, | n |) \qquad (m, n \text{ real numbers})$$

does not carry over to complex numbers. The answer to their question is "no" in as much as (1) does not hold for complex-valued functions in general. To see this let us take for F, f the constant functions

$$F(x) = -1, \qquad f(x) = +1 \qquad (a \leqslant x \leqslant b).$$

If S is the singleton set consisting of the constant function $s = 1 + i$, then the left-hand side of (1) is equal to $2^{1/2} + 1$ whereas the right-hand side is equal to $5^{1/2}$.

However, for complex-valued functions, the following analogous result can be obtained in a straightforward manner.

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THEOREM 1. *Let F, f, and s be complex-valued functions on $[a, b]$. Then*

$$\max(\| F - s \|, \| f - s \|) = \left\| \left\{ \left(\left| \frac{F+f}{2} - s \right| + \frac{|F-f|}{2} \right)^2 + \frac{| |F-s|^2 - |f-s|^2 |}{2} - \frac{|(F-s)^2 - (f-s)^2|}{2} \right\}^{1/2} \right\|. \quad (2)$$

Hence, (2) implies (1) if F, f, and s are real valued.

Let A be a bounded subset of $B[a, b]$, the Banach space of bounded real-valued functions defined on $[a, b]$ with the supremum norm. We have the following

THEOREM 2. *For any $s \in B[a, b]$,*

$$\| |((F+f)/2) - s| + \tfrac{1}{2} |F - f| \| = \sup\{\| a - s \| : a \in A\}, \quad (3)$$

where $F(t) = \sup\{a(t): a \in A\}$ and $f(t) = \inf\{a(t): a \in A\}$.

Hence, Theorem A might have wide applicability, for instance, in computing Chebyshev centers.

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REFERENCE

1. J. B. DIAZ AND H. W. MCLAUGHLIN, On simultaneous Chebyshev approximation and Chebyshev approximation with an additive weight, *J. Approximation Theory* **6** (1972), 68–71.